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On a certain trace of Selberg type

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1 Introduction and statement of the main result

In the spectral theory of automorphic functions, there are two famous trace formulas. One is the Selberg trace formula and the other is the Bruggeman-Kuznetsov trace formula. Researching a relation between these formulas is an interesting problem. For example Joyner [**J**:Section 1], following the idea presented by Zagier, has succeeded in deriving the Bruggeman-Kuznetsov trace formula from the spectral decomposition of the kernel function of invariant integral operator and calculating its $(n, -m)$th Fourier coefficient.

In this article we consider the converse process. That is, starting with the Bruggeman-Kuznetsov trace formula we take a sum over n putting $m=n$ in (1.8) and multiplying $n^{-w}(w\in\mathbf{C})$ on both sides, then we can obtain a trace of the form $\sum_{j\geq 1}L_{j}(w)h(r_{j})$, where $L_{j}(w)$ is the Rankin-Selberg zeta function. This type of sum has already been considered by Zagier [**Z**], more precisely he considers the sum $\sum_{j\geq 1}\tilde{L}_{j}(w)h(r_{j})$(see (1.19)) and its more extended one, and has proved some interesting results by using his formula for the sum(see [**Z**:Theorem 1]). The aim of this article is to give, for the sum $\sum_{j\geq 1}L_{j}(w)h(r_{j})$, an expression which is different from that of Zagier, while we restrict $h(r)$ to a special function as in (1.21). In fact we show that the sum can be expressed in terms of the inner product consisting of the product of the theta series and the non-holmorphic Poincaré series against the Eisenstein series of 1/2-integral weight(see Theorem 1.1). A relation for the Kloosterman sum proved by Kuznetsov [**K**:Theorem 4](see (1.6)) and an expression stated in Proposition 3.1 for the Fourier coefficient of the non-holomorphic Poincaré series play an important role in the proof of Theorem 1.1. In [**M**:Lemma 2.8,2.9] Motohashi obtained a formula for the inner product of the non-holomorphic Poincaré series. By using this we can obtain the expression in Proposition 3.1.

All terms appearing in Theorem 1.1 have a pole at $w=1$. Thus computing their residues we can derive a formula for the sum $\sum_{j\geq 1}\Psi(s,r_{j})=\sum_{j\geq 1}\Gamma(s-\frac{1}{2}-ir)\Gamma(s-\frac{1}{2}+ir)$, which is

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a trace of Selberg type when we take the funcion $\Psi(s,r)$ as the Selberg transform, and is expressed by using the theta series and the non-holomorphic Poincaré series(see Theorem 4.2). This article is a survey of the manuscript [**Y2**].

We first recall the Bruggeman-Kuzunetsov trace formula over the full modular group $\Gamma = PSL(2,\mathbf{Z})$. In this article we always assume that Γ denotes the full modular group $PSL(2,\mathbf{Z})$. Let $\mathcal{H}=\{z=x+iy\in\mathbf{C} \mid y>0\}$ be the complex upper half plane equipped with the hyperbolic measure $d\mu(z) = dxdy/y^2$. Let $L^2(\Gamma\backslash\mathcal{H})$ be the Hilbert space consisting of all functions which are Γ-automorphic and square-integrable for the inner product

$$\langle f(z), g(z)\rangle = \int_{\Gamma\backslash\mathcal{H}} f(z)\overline{g(z)}\, d\mu(z), \tag{1.1}$$

where $\Gamma\backslash\mathcal{H}$ is a fundamental domain of Γ and $\overline{g}$ the complex conjugate of g. Let $\{u_j(z)\}_{j\geq 1}$ be an orthonormal basis of the subspace of all cusp forms in $L^2(\Gamma\backslash\mathcal{H})$. We have the Fourier expansion

$$u_j(z) = \sum_{n\neq 0} \varrho_j(n) y^{\frac{1}{2}} K_{ir_j}(2\pi|n|y)e^{2\pi inx}, \tag{1.2}$$

where $K_\nu(y)$ is the K-Bessel function defined, for example(see [**W**:pp.182,(8)]), by

$$K_\nu(y) = \frac{1}{2}\int_0^\infty e^{-\frac{y}{2}(t+\frac{1}{t})} t^{-\nu-1}\, dt \tag{1.3}$$

for $y>0$ and $\nu\in\mathbf{C}$. Each u_j is an eigenfunction of the Laplacian with eigenvalue $\frac{1}{4}+r_j^2(r_j > 0)$.

Let $\tilde{\Gamma}$ be an arbitrary Fuchsian group of the first kind with a cusp ∞. For $\gamma=\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in\tilde{\Gamma}$ and $z\in\mathcal{H}$, we denote the linear fractional transformation by $\gamma z(:=(az+b)/(cz+d))$ and put $\gamma z = x(\gamma z)+iy(\gamma z)$, that is, $x(\gamma z)$ or $y(\gamma z)$ is real or imaginary part of $\gamma z\in\mathcal{H}$. Let $\Gamma_\infty := \{\left(\begin{smallmatrix} 1 & \ell\\ & 1\end{smallmatrix}\right) \mid \ell\in\mathbf{Z}\}$ be the stability subgroup in $\Gamma(=PSL(2,\mathbf{Z}))$ of a cusp at infinity. For $z\in\mathcal{H}$ and $s\in\mathbf{C}$, the Eisenstein series for the group Γ is defined by

$$E(z,s,\Gamma) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} y(\gamma z)^s. \tag{1.4}$$

This series converges absolutely and uniformly for $\Re(s)>1$ and it is well known that the series can be continued meromorphically to the whole complex s-plane by using its Fourier expansion.

For arbitrary nonzero integers m,n, the Kloosterman sum for the group Γ is defined by

$$S(m,n,c,\Gamma) = \sum_{0\leq a,d<c} e^{2\pi i\frac{ma+nd}{c}}, \tag{1.5}$$

where the sum is taken over the elements $\binom{a\ *}{c\ d} \in \Gamma$ for any fixed $c > 0$. In the paper [**K**:Theorem 4] Kuznetsov proved the following relation

$$S(m,n,c,\Gamma) = \sum_{d|(m,n,c)} dS(1, \frac{mn}{d^2}, \frac{c}{d}, \Gamma). \tag{1.6}$$

This relation plays an important role in the proof of Theorem 1.1.

Let ν be a complex variable. Then the Bessel function J_ν is defiend by

$$J_\nu(y) = \pi^{-1/2} \frac{1}{\Gamma(\nu+\frac{1}{2})} \Big(\frac{z}{2}\Big)^\nu \int_{-1}^{1} e^{iyt}(1-t^2)^{\nu-\frac{1}{2}}\, dt \tag{1.7}$$

for $\Re(\nu) > -1/2$ and $y > 0$(see [**W**:pp.48,(4),pp.172,(2)]). The modified Bessel function I_ν has an expression with e^{-yt} instead of e^{iyt} in (1.7). Under these notations the Bruggeman-Kuznetsov trace formula is stated as follows.

The Bruggeman-Kuznetsov trace formula. *Let m, n be nonzero integers. Let $h(r)$ be a function of a complex variable r satisfying certain conditions. Then*

$$\begin{aligned} &\sum_{j\geq 1} \frac{\overline{\varrho_j(m)}\varrho_j(n)}{\cosh(\pi r_j)} h(r_j) \\ &\quad + \frac{1}{\pi}\int_{-\infty}^{\infty} \Big|\frac{n}{m}\Big|^{ir} \frac{\sigma_{2ir}(|m|)\sigma_{-2ir}(|n|)}{\zeta(1-2ir)\zeta(1+2ir)} h(r)\, dr \\ &= \frac{\delta_{m,n}}{\pi^2}\int_{-\infty}^{\infty} r\tanh(\pi r) h(r)\, dr \\ &\quad + \sum_{c=1}^{\infty} \frac{S(m,n,c,\Gamma)}{c} \frac{2i}{\pi} \int_{-\infty}^{\infty} r M_{2ir}(4\pi\frac{|mn|^{\frac{1}{2}}}{c}) \frac{h(r)}{\cosh(\pi r)}\, dr, \end{aligned} \tag{1.8}$$

where the sum over j runs over the eigenvalues of the space of cusp forms in $L^2(\Gamma\backslash\mathcal{H})$, $\zeta(*)$ is the Riemann zeta function, $\delta_{m,n}$ is the Kronecker symbol, $\sigma_\nu(|n|)$ is the sum of the νth powers of divisors of $|n|$, and M_ν stands for the Bessel function J_ν or the modified Bessel function I_ν according as $mn > 0$ or $mn < 0$.

This formula was first proved by Kuznetsov [**K**], and a little later by Bruggeman [**B1**] and [**B2**]. Let ε and δ be arbitrarily small positive constants. Kuznetsov states the formula (1.8) for the class of functions $h(r)$ which are even and holomorphic in the strip $|\Im(r)| < \frac{1}{2}+\epsilon$, and $|h(r)| \ll (1+|r|)^{-2-\delta}$ as $|r| \to \infty$. On the other hand Bruggeman, for $h(r)$ which are even and holomorphic for $|\Im(r)| < \frac{1}{4}+\varepsilon$ and satisfy the same decrease condition as that of Kuznetsov. Thus Bruggeman's result permits more wide class of $h(r)$ than that of Kuznetsov. For other proofs of (1.8) different from those of Bruggeman and Kuznetsov, the reader is referred to Iwaniec [**I**], Motohashi [**M**] and Joyner [**J**].

For $m \in \mathbf{Z}_{\neq 0}$, $z \in \mathcal{H}$ and $s \in \mathbf{C}$, the non-holomorphic Poincaré series for the group Γ is defined by

$$P_m(z,s,\Gamma) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} e^{2\pi i m x(\gamma z)} e^{-2\pi |m| y(\gamma z)} y(\gamma z)^s. \tag{1.9}$$

This converges absolutely and uniformly for $\Re(s) > 1$ and belongs to the Hilbert space $L^2(\Gamma \backslash \mathcal{H})$. This series appears in Theorem 1.1 as an important constituent.

For a complex number $z \neq 0$ we define the power $z^{1/2}$ by $z^{1/2} = |z|^{1/2} \exp(\frac{1}{2} i \arg z)$ with $-\pi < \arg z \leq \pi$. Let $\Gamma_0(N) (\subset SL(2,\mathbf{Z}))$ be the Hecke congruence group of level N:

$$\Gamma_0(N) = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbf{Z}) | c \equiv 0 \pmod N \}.$$

Then for $z \in \mathcal{H}$ the theta series is defined by

$$\Theta(z) = \sum_{n=-\infty}^{\infty} e^{2\pi i n^2 z}, \tag{1.10}$$

and the theta multiplier, by

$$j(\gamma, z) = \Theta(\gamma z)/\Theta(z) \tag{1.11}$$

for $\gamma \in \Gamma_0(4)$. It is known that

$$j(\gamma, z) = \left(\frac{c}{d}\right) \varepsilon_d^{-1} (cz+d)^{1/2}, \tag{1.12}$$

where $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(4)$, $\varepsilon_d = 1$ or i according as $d \equiv 1$ or $3 \pmod 4$, and (c/d) is the extended Legendre symbol. For precise definition of (c/d), see [**Sh2**].

Shimura [**Sh1**:(1.4)] introduced the Eisenstein series of half-integral weight(see (2.1)). Following this we define the Eisenstein series $E_{1/2}$ of 1/2-integral weight as follows

$$E_{1/2}(z,w) = E_{1/2}(z,w,\Gamma_0(4)) := \sum_{\gamma \in \widehat{\Gamma}_\infty \backslash \Gamma_0(4)} j(\gamma, z) y(\gamma z)^w, \tag{1.13}$$

where $w \in \mathbf{C}$ and $\widehat{\Gamma}_\infty := \{ \pm \left(\begin{smallmatrix} 1 & \ell \\ & 1 \end{smallmatrix}\right) | \ell \in \mathbf{Z} \}$. This series converges absolutely and uniformly for $\Re(w) > 1 + \frac{1}{4}$. It has been proved by Shimura [**Sh1**] that the series can be continued meromorphically to the whole complex w-plane.

For $s, r \in \mathbf{C}$ we define the function $\Psi(s,r)$ by

$$\Psi(s,r) = \Gamma(s - \frac{1}{2} - ir)\Gamma(s - \frac{1}{2} + ir). \tag{1.14}$$

Let w, s be complex variables. Then we denote by $J(w,s)$ the following integral

$$J(w,s)=\int_{-\infty}^{\infty}\frac{\zeta(w-2ir)\zeta(w+2ir)}{\zeta(1-2ir)\zeta(1+2ir)}\cosh(\pi r)\Psi(s,r) \times\Gamma(\frac{w}{2}-ir)\Gamma(\frac{w}{2}+ir)\,dr. \tag{1.15}$$

Let C_0 be a deformation in the complex r-plane of the real axis into the strip $0<\Im(r)$ which is sufficiently close to the real axis that all zeros of the Riemann zeta function lie to the left of $1+2iC_0$ and $\zeta(1+2ir)=O(|r|^{\varepsilon})$ for $r\in C_0$. Moreover we define

$$J_{C_0}(w,s)=\int_{C_0}\frac{\zeta(w-2ir)\zeta(w+2ir)}{\zeta(1-2ir)\zeta(1+2ir)}\cosh(\pi r)\Psi(s,r) \times\Gamma(\frac{w}{2}-ir)\Gamma(\frac{w}{2}+ir)\,dr. \tag{1.16}$$

Let U be the domain, in the complex w-plane, enclosed by $1+2iC_0$ and $1-2iC_0$. Then we define the function $H(w,s)$ as follows assuming $\Re(s)>1$ for simplicity:

$$H(w,s)=\begin{cases} J(w,s) & \text{for } \Re(w)>1,\\ J_{C_0}(w,s)+D(w,s) & \text{for } w\in U,\\ J(w,s)+2D(w,s) & \text{for } 0<\Re(w)<1,\end{cases} \tag{1.17}$$

where

$$D(w,s)=\pi\frac{\zeta(2w-1)}{\zeta(w)\zeta(2-w)}\cosh(\pi\frac{1-w}{2i})\Psi(s,\frac{1-w}{2i})\Gamma(w-\frac{1}{2})\Gamma(\frac{1}{2}). \tag{1.18}$$

Let $L_j(w):=\sum_{n=1}^{\infty}|\varrho_j(n)|^2n^{-w}$ be the Rankin-Selberg L-function, and put

$$\widetilde{L}_j(w):=\Gamma(\frac{w}{2}-ir_j)\Gamma(\frac{w}{2}+ir_j)L_j(w). \tag{1.19}$$

It is known that the function $\widetilde{L}_j(w)$ has the expression

$$\widetilde{L}_j(w)=2^2\pi^w\frac{\Gamma(w)}{\Gamma^2(\frac{w}{2})}\int_{\Gamma\backslash\mathcal{H}}|u_j(z)|^2E(z,w,\Gamma)\,d\mu(z), \tag{1.20}$$

where $u_j(z)$ is the Maass cusp form defined by (1.2) and $E(z,w,\Gamma)$ is the Eisenstein series as in (1.4). By the right-hand side the function $\widetilde{L}_j(w)$ can be continued meromorphically to the whole complex w-plane.

We shall state the main result in this article. In the Bruggeman-Kuznetsov trace formula (1.8) we adopt the following function as $h(r)$:

$$\Psi(s,r)\Gamma(\frac{w}{2}-ir)\Gamma(\frac{w}{2}+ir)\cosh(\pi r). \tag{1.21}$$

This satisfies Kuznetsov's condition if $\Re(s)$, $\Re(w)>1$. Putting $m=n$ and multiplying n^{-w} on both sides of (1.8), we take a sum over $n(\geq 1)$. Then we show that

THEOREM 1.1. *We have the equality*

$$\begin{aligned}\sum_{j\geq 1}\widetilde{L}_j(w)\Psi(s,r_j) &= 2^w\pi^{\frac{w}{2}-1}(4\pi)^{s-\frac{1}{2}}\Gamma(\frac{w}{2}+\frac{1}{2})\Gamma(s) \\ &\times\zeta(w)\frac{1}{2}\int_{\Gamma_0(4)\backslash\mathcal{H}}\overline{\Theta(z)}P_1(z,s,\Gamma)\overline{E_{1/2}(z,\frac{\bar{w}}{2}+\frac{1}{2})}\,d\mu(z) \\ &-2^w\pi^w\Gamma(s+\frac{w}{2}-\frac{1}{2})\Gamma(s-\frac{w+1}{2})\frac{\zeta(w)}{\zeta(w+1)} \\ &-\frac{\zeta^2(w)}{\zeta(2w)}\frac{1}{\pi}H(w,s)\end{aligned} \tag{1.22}$$

for $\Re(w)\geq 1/2$ *and* $\Re(s)>\max(1,\Re(\frac{w+1}{2}))$, *where* $\Psi(s,r)$ *is the function defined by* (1.14), $\overline{\Theta(z)}$ *is the complex conjugate of the theta series,* $E_{1/2}$ *is the Eisenstein series of* $1/2$*-integral weight defined by* (1.13) *and its meromorphic continuation,* P_1 *is the non-holomorphic Poincaré series defined by* (1.9), *and* $H(w,s)$ *is the function defined by* (1.17) *and* (1.18).

2 Eisenstein series of half-integral weight

The Eisenstein series of half-integral weight was introduced by Shimura[**Sh1**], and he has established basic properties of the series. In this section we especially recall the Fourier expansion of the series.

Let k be an odd(positive or negative) integer, ω an arbitrary character modulo N, and N a multiple of 4. Moreover let W be a set of representatives for $\widehat{\Gamma}_\infty\backslash\Gamma_0(N)$. Assume that $\omega(-1)=1$. Then for $z=x+iy\in\mathcal{H}$ and $s\in\mathbf{C}$, Shimura [**Sh1**:(1.4)] introduced the Eisenstein series $E(z,s)$ of half-integral weight as follows:

$$\begin{aligned}E(z,s) &= E(z,s,k,\omega) \\ &= y^{s/2}\sum_{\gamma\in W}\omega(d_\gamma)j(\gamma,z)^k|j(\gamma,z)|^{-2s},\end{aligned} \tag{2.1}$$

where d_γ is the lower right entry of γ, and $j(\gamma,z)$ is that of (1.12). This series converges absolutely and uniformly for $\Re(s)>(k+4)/2$.

We shall recall the Fourier expansion of $E(z,s)$. Let us introduce a confluent hypergeometric function $\sigma(y,\alpha,\beta)$ by

$$\sigma(y,\alpha,\beta) = \int_0^\infty (u+1)^{\alpha-1} u^{\beta-1} e^{-yu}\, du, \tag{2.2}$$

where y is real positive, and α,β are complex variables. This is convergent for $\Re(\beta) > 0$. Moreover using $\sigma(y,\alpha,\beta)$ we define the function $\tau_n(y,\alpha,\beta)$ by

$$\begin{aligned} & i^{\alpha-\beta}(2\pi)^{-\alpha-\beta}\Gamma(\alpha)\Gamma(\beta)\tau_n(y,\alpha,\beta) \\ &= \begin{cases} n^{\alpha+\beta-1}e^{-2\pi n y}\sigma(4\pi n y,\alpha,\beta) & \text{for } n>0\ , \\ |n|^{\alpha+\beta-1}e^{-2\pi|n|y}\sigma(4\pi|n|y,\beta,\alpha) & \text{for } n<0, \\ \Gamma(\alpha+\beta-1)(4\pi y)^{1-\alpha-\beta} & \text{for } n=0. \end{cases} \end{aligned} \tag{2.3}$$

We denote by $E'(z,s)$ the following quantity:

$$E'(z,s) = E'(z,s,k,\omega) = E(-1/Nz,s)(-iz\sqrt{N})^{k/2}.$$

Then, Shimura [**Sh1**:(3.2),(3.3)] states the Fourier expansion

$$\begin{aligned} & N^{(2s-k)/4} i^{k/2} y^{-s/2} E'(z,s) \\ &= \sum_{n=-\infty}^{\infty} \alpha(n,s) e^{2\pi i n x} \tau_n(y, \frac{s-k}{2}, \frac{s}{2}). \end{aligned}$$

Therefore we have

$$E(-\frac{1}{Nz},s,k,\omega) = N^{-\frac{s}{2}} z^{-\frac{k}{2}} y^{\frac{s}{2}} \sum_{n=-\infty}^{\infty} \alpha(n,s) e^{2\pi i n x} \tau_n(y,\frac{s-k}{2},\frac{s}{2}). \tag{2.4}$$

For a character ω, we put $L(s,\omega) = \sum_{n=1}^{\infty} \omega(n) n^{-s}$. To emphasize the possible missing factors, we also write $L_N(s,\omega)$ for $L(s,\omega)$, thus $L_N(s,\omega) = \sum_{(n,N)=1} \omega(n) n^{-s}$. As for the term $\alpha(n,s)$ in (2.4), Shimura [**Sh1**:Proposition 1] states as follows. Let t be a (positive or negative) square-free integer. Putting $\lambda = (k+1)/2$ we define the characters ω_1 and ω_2 by

$$\begin{aligned} \omega_1(a) &= \left(\frac{-1}{a}\right)^{\lambda} \left(\frac{tN}{a}\right) \omega(a) && \text{for } (a,tN)=1\ , \\ \omega_2(a) &= \omega(a)^2 && \text{for } (a,N)=1. \end{aligned} \tag{2.5}$$

Then for $n = tm^2$ with a positive integer m, we have

$$\begin{aligned} & L_N(2s-2\lambda,\omega_2)\alpha(n,s) = L_N(s-\lambda,\omega_1)\beta(n,s), \\ & \beta(n,s) = \sum \mu(a)\omega_1(a)\omega_2(b) a^{\lambda-s} b^{k+2-2s}, \end{aligned} \tag{2.6}$$

where the last sum is extended over all positive integers a, b prime to N such that ab divides m, and μ denotes the Möbius function. Furthermore for $n=0$,

$$\alpha(0,s) = L_N(2s-k-2,\omega_2)/L_N(2s-2\lambda,\omega_2). \tag{2.7}$$

The series $E_{1/2}$ defined by (1.13) is a special case of E in (2.1). In fact we have only to put $k=1$, $N=4$ and ω being principal. However it should be noted that $E_{1/2}(z,s) = E(z,2s)$. Substituting these facts into (2.4) we have the expansion

$$\begin{aligned} E_{1/2}(-\frac{1}{4z},\frac{w}{2}+\frac{1}{2}) &= 4^{-\frac{w+1}{2}}z^{-\frac{1}{2}}y^{\frac{w+1}{2}} \\ &\times \sum_{n=-\infty}^{\infty} \alpha(n,w+1)e^{2\pi inx}\tau_n(y,\frac{w}{2},\frac{w+1}{2}). \end{aligned} \tag{2.8}$$

Here since $\lambda=1$ and $N=4$ the character ω_1 in (2.5) is equal to

$$\omega_1(a) = \left(\frac{-4t}{a}\right) \quad \text{for } (a,4t)=1\ , \tag{2.9}$$

and ω_2 is a principal character modulo 4. Denoting $\zeta_4(w) = 1^{-w}+3^{-w}+5^{-w}\cdots$, the term $\alpha(n,w+1)$ in (2.6) is described as

$$\begin{aligned} \alpha(n,w+1) &= \frac{L_4(w,\omega_1)}{\zeta_4(2w)}\beta(n,w+1), \\ \beta(n,w+1) &= \sum \mu(a)\omega_1(a)a^{1-(w+1)}b^{1+2-2(w+1)} \end{aligned} \tag{2.10}$$

for $n=tm^2$, where the last sum is extended over all positive integers a, b prime to 4 such that ab divides m. Moreover

$$\alpha(0,w+1) = \frac{\zeta_4(2w-1)}{\zeta_4(2w)}. \tag{2.11}$$

Notice that the character ω_1 in (2.9) turns out to be principal for $t=-1$. Therefore substituting $-1/4z$ for z in (2.8) we obtain the following expansion:

$$\begin{aligned} E_{1/2}(z,\frac{w}{2}+\frac{1}{2}) &= i^{\frac{1}{2}}2^{\frac{1}{2}-2w}\pi\frac{\Gamma(w-\frac{1}{2})}{\Gamma(\frac{w}{2})\Gamma(\frac{w+1}{2})}2(-i)z^{\frac{1}{2}}Y^{1-\frac{w}{2}}\frac{\zeta_4(2w-1)}{\zeta_4(2w)} \\ &+4^{-\frac{w+1}{2}}2(-i)z^{\frac{1}{2}}Y^{\frac{w+1}{2}}\frac{\zeta_4(w)}{\zeta_4(2w)} \\ &\times \sum_{m=1}^{\infty}\beta(-m^2,w+1)e^{-2\pi im^2X}\tau_{-m^2}(Y,\frac{w}{2},\frac{w+1}{2}) \\ &+4^{-\frac{w+1}{2}}2(-i)z^{\frac{1}{2}}Y^{\frac{w+1}{2}}\frac{1}{\zeta_4(2w)} \\ &\times\sum_t\sum_{m=1}^{\infty}L_4(w,\omega_1)\beta(tm^2,w+1)e^{2\pi itm^2X}\tau_{tm^2}(Y,\frac{w}{2},\frac{w+1}{2}), \end{aligned} \tag{2.12}$$

where $Y=y/4(x^2+y^2)$ and $X=-x/4(x^2+y^2)$, and the summand t runs over all positive($t\geq 1$) and negative($t\leq -2$) square-free integers.

Shimura has already established the convergence of the series on the right-hand side of (2.4), and meromorphic continuation of $E(-1/Nz,s,k,\omega)$ to the whole complex s-plane. Thus the series $E_{1/2}$ in (2.8)(or (2.12)) is also continued to the whole complex w-plane.

3 Outline of the proof of Theorem 1.1

In this section we give an outline of the proof of Theorem 1.1. For precise proof the reader is referred to [**Y2**].

Recall the Bruggeman-Kuznetsov trace formula (1.8), and adopt the function in (1.21) as $h(r)$. Then putting $m=n$ and multiplying n^{-w} on both sides, we take a sum over n; that is we consider the following quantity:

$$\begin{aligned}
&\sum_{n=1}^{\infty}\frac{1}{n^w}\sum_{j\geq 1}|\varrho_j(n)|^2\Gamma(\frac{w}{2}-ir_j)\Gamma(\frac{w}{2}+ir_j)\Psi(s,r_j) \qquad (3.1)\\
&\quad=\sum_{n=1}^{\infty}\frac{1}{n^w}\sum_{c=1}^{\infty}\frac{S(n,n,c,\Gamma)}{c}\frac{2i}{\pi}\int_{-\infty}^{\infty}rJ_{2ir}(4\pi\frac{n}{c})\\
&\qquad\qquad\times\Psi(s,r)\Gamma(\frac{w}{2}-ir)\Gamma(\frac{w}{2}+ir)\,dr\\
&\quad+\sum_{n=1}^{\infty}\frac{1}{n^w}\frac{1}{\pi^2}\int_{-\infty}^{\infty}r\sinh(\pi r)\Psi(s,r)\Gamma(\frac{w}{2}-ir)\Gamma(\frac{w}{2}+ir)\,dr\\
&\quad-\sum_{n=1}^{\infty}\frac{1}{n^w}\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sigma_{2ir}(n)\sigma_{-2ir}(n)}{\zeta(1-2ir)\zeta(1+2ir)}\\
&\qquad\qquad\times\cosh(\pi r)\Psi(s,r)\Gamma(\frac{w}{2}-ir)\Gamma(\frac{w}{2}+ir)\,dr.
\end{aligned}$$

We can state absolute convergence of each term at least for $\Re(w)>2$ and $\Re(s)>1$.

Since we denote $L_j(w):=\sum_{n=1}^{\infty}|\varrho_j(n)|^2n^{-w}$ and $\tilde{L}_j(w):=\Gamma(\frac{w}{2}-ir_j)\Gamma(\frac{w}{2}+ir_j)L_j(w)$(see (1.19)), the left-hand side can be described as

$$\sum_{j\geq 1}\tilde{L}_j(w)\Psi(s,r_j). \qquad (3.2)$$

Here by using the expression (1.20) we can continue the sum (3.2) to the whole complex w,s-plane which has a simple pole at $w=1$.

The second term on the right-hand side of (3.1) is

$$\zeta(w)\frac{1}{\pi^2}\int_{-\infty}^{\infty}r\sinh(\pi r)\Psi(s,r)\Gamma(\frac{w}{2}-ir)\Gamma(\frac{w}{2}+ir)\,dr. \qquad (3.3)$$

Moreover since we know the formula

$$\sum_{n=1}^{\infty}\frac{|\sigma_{ir}(n)|^2}{n^w}=\frac{\zeta^2(w)\zeta(w-ir)\zeta(w+ir)}{\zeta(2w)},$$

the third term on the right-hand side of (3.1) turns out to be equal to

$$\begin{aligned}&-\frac{\zeta^2(w)}{\zeta(2w)}\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\zeta(w-2ir)\zeta(w+2ir)}{\zeta(1-2ir)\zeta(1+2ir)}\\&\qquad\qquad\times\cosh(\pi r)\Psi(s,r)\Gamma(\frac{w}{2}-ir)\Gamma(\frac{w}{2}+ir)\,dr\\&=-\frac{\zeta^2(w)}{\zeta(2w)}\frac{1}{\pi}J(w,s)\end{aligned}\tag{3.4}$$

recalling the definition of $J(w,s)$ as in (1.15). Therefore by defining the function $H(w,s)$ as in (1.17) and (1.18), we can continue the third term to the domain $\Re(w)>0$. Concerning this argument, see Zagier [**Z**:pp.335-337].

From now on, we denote the first term on the right-hand side of (3.1) by I, and transform it into an interesting form. First using the relation (1.6) for the Kloosterman sum and putting $\ell:=c/d$ we have

$$\begin{aligned}I=\sum_{n=1}^{\infty}\frac{1}{n^w}\sum_{d|n}\sum_{\ell=1}^{\infty}\frac{S(1,(\frac{n}{d})^2,\ell,\Gamma)}{\ell}\frac{2i}{\pi}\int_C rJ_{2ir}(4\pi\frac{n}{d}\frac{1}{\ell})\\\times\Psi(s,r)\Gamma(\frac{w}{2}-ir)\Gamma(\frac{w}{2}+ir)\,dr.\end{aligned}$$

It is equal to

$$\begin{aligned}&\sum_{\ell=1}^{\infty}\frac{1}{\ell}\frac{2i}{\pi}\int_{-\infty}^{\infty}r\Psi(s,r)\Gamma(\frac{w}{2}-ir)\Gamma(\frac{w}{2}+ir)\\&\qquad\times\Big\{\sum_{n=1}^{\infty}\frac{1}{n^w}\sum_{d|n}S(1,(\frac{n}{d})^2,\ell,\Gamma)J_{2ir}(4\pi\frac{n}{d}\frac{1}{\ell})\Big\}\,dr\\&=\sum_{\ell=1}^{\infty}\frac{1}{\ell}\frac{2i}{\pi}\int_{-\infty}^{\infty}r\Psi(s,r)\Gamma(\frac{w}{2}-ir)\Gamma(\frac{w}{2}+ir)\\&\qquad\times\zeta(w)\sum_{n=1}^{\infty}\frac{S(1,n^2,\ell,\Gamma)}{n^w}J_{2ir}(4\pi\frac{n}{\ell})\,dr.\end{aligned}$$

Thus we have

$$\begin{aligned}I=\zeta(w)\sum_{n=1}^{\infty}\frac{1}{n^w}\sum_{\ell=1}^{\infty}\frac{S(1,n^2,\ell,\Gamma)}{\ell}\frac{2i}{\pi}\int_{-\infty}^{\infty}rJ_{2ir}(4\pi\frac{n}{\ell})\\\times\Psi(s,r)\Gamma(\frac{w}{2}-ir)\Gamma(\frac{w}{2}+ir)\,dr.\end{aligned}\tag{3.5}$$

Here we apply the formula

$$\frac{1}{n^w}\Gamma(\frac{w}{2}-ir)\Gamma(\frac{w}{2}+ir)$$
$$=\Gamma(\frac{w}{2}+\frac{1}{2})2^w\pi^{\frac{w}{2}-\frac{1}{2}}\int_0^\infty y^{\frac{1}{2}}K_{ir}(2\pi n^2y)y^{\frac{w}{2}-\frac{3}{2}}e^{-2\pi n^2 y}\,dy.$$

Therefore we obtain

$$I=2^w\pi^{\frac{w}{2}-\frac{1}{2}}\Gamma(\frac{w}{2}+\frac{1}{2}) \tag{3.6}$$
$$\times\zeta(w)\int_0^\infty y^{\frac{w}{2}-\frac{3}{2}}\,dy\cdot\sum_{n=1}^\infty e^{-2\pi n^2 y}$$
$$\times\Big(\sum_{\ell=1}^\infty\frac{S(1,n^2,\ell,\Gamma)}{\ell}\frac{2i}{\pi}\int_C rJ_{2ir}(4\pi\frac{n}{\ell})\Psi(s,r)y^{\frac{1}{2}}K_{ir}(2\pi n^2y)\,dr\Big).$$

To proceed further we prepare the following proposition. Let $P_m(z,s,\Gamma)$ be the non-holomorphic Poincaré series defined by (1.9), and let $a_m(y,s,n,\Gamma)$ be the nth Fourier coefficient of the series P_m:

$$a_m(y,s,n,\Gamma)=\int_0^1 P_m(x+iy,s,\Gamma)e^{-2\pi inx}\,dx.$$

Then we have the following

PROPOSITION 3.1. *Let m, n be nonzero integers, and s a complex number. For $\Re(s)>1$ we have*

$$a_m(y,s,n,\Gamma)=\pi^{\frac{1}{2}}(4\pi|m|)^{\frac{1}{2}-s}\frac{1}{\Gamma(s)} \tag{3.7}$$
$$\times\Big\{\frac{\delta_{m,n}}{\pi^2}\int_{-\infty}^\infty r\sinh(\pi r)\Psi(s,r)y^{\frac{1}{2}}K_{ir}(2\pi|n|y)\,dr$$
$$+\sum_{c=1}^\infty\frac{S(m,n,c,\Gamma)}{c}\frac{2i}{\pi}\int_{-\infty}^\infty rM_{2ir}(4\pi\frac{|mn|^{\frac{1}{2}}}{c})\Psi(s,r)y^{\frac{1}{2}}K_{ir}(2\pi|n|y)\,dr\Big\},$$

where $\Gamma=PSL(2,\mathbf{Z})$, M_{2ir} is as in (1.8), *and Ψ is that of* (1.14).

In [**M**: Lemma 2.8, 2.9], Motohashi obtained a formula for the inner product of the non-holomorphic Poincaré series. Based on his formula we can derive the expression (3.7). In view of this we have

$$a_1(y,s,n^2,\Gamma)=\pi^{\frac{1}{2}}(4\pi)^{\frac{1}{2}-s}\frac{1}{\Gamma(s)}$$
$$\times\Big\{\frac{\delta_{1,n^2}}{\pi^2}\int_{-\infty}^\infty r\sinh(\pi r)\Psi(s,r)y^{\frac{1}{2}}K_{ir}(2\pi n^2y)\,dr$$
$$+\sum_{\ell=1}^\infty\frac{S(1,n^2,\ell,\Gamma)}{\ell}\frac{2i}{\pi}\int_{-\infty}^\infty rJ_{2ir}(4\pi\frac{n}{\ell})\Psi(s,r)y^{\frac{1}{2}}K_{ir}(2\pi n^2y)\,dr\Big\}.$$

Substituting this into (3.6) we obtain

$$\begin{aligned}
I &= 2^w \pi^{\frac{w}{2}-1}(4\pi)^{s-\frac{1}{2}}\Gamma(\frac{w}{2}+\frac{1}{2})\Gamma(s) \\
&\times\zeta(w)\int_0^\infty \Big(\sum_{n=1}^\infty e^{-2\pi n^2 y} a_1(y,s,n^2,\Gamma)\Big)\, y^{\frac{w}{2}-\frac{3}{2}}\,dy \\
&-2^w\pi^{\frac{w}{2}-\frac{1}{2}}\Gamma(\frac{w}{2}+\frac{1}{2}) \\
&\times\zeta(w)\int_0^\infty y^{\frac{w}{2}-\frac{3}{2}} e^{-2\pi y}dy\cdot\frac{1}{\pi^2}\int_{-\infty}^\infty r\sinh(\pi r)\Psi(s,r) y^{\frac{1}{2}} K_{ir}(2\pi y)\,dr.
\end{aligned}$$

Since

$$\int_0^\infty e^{-2\pi y} y^{\frac{1}{2}} K_{ir}(2\pi y) y^{\frac{w}{2}-\frac{3}{2}}\,dy = (2\pi)^{-\frac{w}{2}}\pi^{\frac{1}{2}}2^{-\frac{w}{2}}\frac{\Gamma(\frac{w}{2}-ir)\Gamma(\frac{w}{2}+ir)}{\Gamma(w+\frac{1}{2})},$$

we can transform the second term further, deriving

$$\begin{aligned}
I &= 2^w \pi^{\frac{w}{2}-1}(4\pi)^{s-\frac{1}{2}}\Gamma(\frac{w}{2}+\frac{1}{2})\Gamma(s) \\
&\times\zeta(w)\int_0^\infty \Big(\sum_{n=1}^\infty e^{-2\pi n^2 y} a_1(y,s,n^2,\Gamma)\Big)\, y^{\frac{w}{2}-\frac{3}{2}}\,dy \\
&-\zeta(w)\frac{1}{\pi^2}\int_{-\infty}^\infty r\sinh(\pi r)\Psi(s,r)\Gamma(\frac{w}{2}-ir)\Gamma(\frac{w}{2}+ir)\,dr.
\end{aligned} \tag{3.8}$$

We continue the transformation. First we have

$$\begin{aligned}
&\sum_{n=1}^\infty e^{-2\pi n^2 y} a_1(y,s,n^2,\Gamma) \\
&= \frac{1}{2}\int_0^1 \overline{\Theta(z)} P_1(z,s,\Gamma)\,dx - \frac{1}{2}a_1(y,s,0,\Gamma).
\end{aligned} \tag{3.9}$$

For the constant term $a_1(y,s,0,\Gamma)$ we know the following formula(see [**Y1**:Theorem B]):

$$\begin{aligned}
a_1(y,s,0,\Gamma) &= 2^{3-2s}\frac{\pi}{\Gamma(s)}\pi^{\frac{1-s}{2}}\sum_{c=1}^\infty \frac{S(1,0,c,\Gamma)}{c^{1+s}} \\
&\times y^s\int_0^\infty e^{-yt^2} t^{3s-2} J_{s-1}(2\pi^{\frac{1}{2}}\frac{t}{c})\,dt.
\end{aligned} \tag{3.10}$$

Then since

$$\int_0^\infty t^{\mu-1}J_\nu(at)\,dt = \frac{2^{\mu-1}\Gamma(\frac{\mu+\nu}{2})}{a^\mu\Gamma(\frac{\nu-\mu}{2}+1)},$$

(see [**W**:pp.391,(1)]), and since $S(1,0,c,\Gamma)=\mu(c)$(Möbius function), we derive that

$$-\frac{1}{2}\int_0^\infty a_1(y,s,0,\Gamma)y^{\frac{w}{2}-\frac{3}{2}}\,dy \tag{3.11}$$

$$= -2^{1-2s}\frac{1}{\Gamma(s)}\pi^{\frac{3+w}{2}-s}\Gamma(s+\frac{w}{2}-\frac{1}{2})\frac{\Gamma(s-\frac{w+1}{2})}{\Gamma(\frac{w+1}{2})}\sum_{c=1}^\infty\frac{\mu(c)}{c^{1+w}}$$

$$= -2^{1-2s}\pi^{\frac{3+w}{2}-s}\frac{\Gamma(s+\frac{w}{2}-\frac{1}{2})\Gamma(s-\frac{w+1}{2})}{\Gamma(s)\Gamma(\frac{w+1}{2})}\frac{1}{\zeta(1+w)}.$$

Therefore gathering (3.8), (3.9) and (3.11) together, we obtain

$$I = 2^w\pi^{\frac{w}{2}-1}(4\pi)^{s-\frac{1}{2}}\Gamma(\frac{w}{2}+\frac{1}{2})\Gamma(s) \tag{3.12}$$

$$\times\zeta(w)\frac{1}{2}\int_0^\infty\int_0^1\overline{\Theta(z)}P_1(z,s,\Gamma)\,dx\,y^{\frac{w}{2}-\frac{3}{2}}\,dy$$

$$-2^w\pi^{\frac{w}{2}-1}(4\pi)^{s-\frac{1}{2}}2^{1-2s}\pi^{\frac{3+w}{2}-s}\Gamma(s+\frac{w}{2}-\frac{1}{2})\Gamma(s-\frac{w+1}{2})\frac{\zeta(w)}{\zeta(w+1)}$$

$$-\zeta(w)\frac{1}{\pi^2}\int_{-\infty}^\infty r\sinh(\pi r)\Psi(s,r)\Gamma(\frac{w}{2}-ir)\Gamma(\frac{w}{2}+ir)\,dr.$$

Finally applying the Rankin-Selberg method to the first term on the right-hand side above we have

$$\int_0^\infty\int_0^1\overline{\Theta(z)}P_1(z,s,\Gamma)\,dx\,y^{\frac{w}{2}-\frac{3}{2}}\,dy$$

$$= \int_{\mathcal{F}}\overline{\Theta(z)}P_1(z,s,\Gamma)\sum_{\gamma\in W}\overline{j(\gamma,z)y(\gamma z)^{\frac{w}{2}+\frac{1}{2}}}\,d\mu(z)$$

$$= \int_{\mathcal{F}}\overline{\Theta(z)}P_1(z,s,\Gamma)\overline{E_{1/2}(z,\frac{\overline{w}}{2}+\frac{1}{2})}\,d\mu(z),$$

where $E_{1/2}(z,\cdot)$ is the Eisenstein series of 1/2-integral weight defined by (1.13)(or (2.12)), and $\mathcal{F}$ is a fundamental domain of $\Gamma_0(4)$. Hence we conclude the expression

$$I = 2^w\pi^{\frac{w}{2}-1}(4\pi)^{s-\frac{1}{2}}\Gamma(\frac{w}{2}+\frac{1}{2})\Gamma(s) \tag{3.13}$$

$$\times\zeta(w)\frac{1}{2}\int_{\mathcal{F}}\overline{\Theta(z)}P_1(z,s,\Gamma)\overline{E_{1/2}(z,\frac{\overline{w}}{2}+\frac{1}{2})}\,d\mu(z)$$

$$-2^w\pi^w\Gamma(s+\frac{w}{2}-\frac{1}{2})\Gamma(s-\frac{w+1}{2})\frac{\zeta(w)}{\zeta(w+1)}$$

$$-\zeta(w)\frac{1}{\pi^2}\int_{-\infty}^\infty r\sinh(\pi r)\Psi(s,r)\Gamma(\frac{w}{2}-ir)\Gamma(\frac{w}{2}+ir)\,dr.$$

By careful estimation we can state the convergence of the inner product above in $\Re(w)\geq 1/2$ and $\Re(s)>\Re(\frac{w}{2}+\frac{1}{2})$. Noticing that the third term in (3.13) and the second term in (3.1)(or (3.3)) cancel, we complete the proof of Theorem 1.1.

4 A trace of Selberg type

In Theorem 1.1, both sides determine meromorphic functions of w, and each term has a pole at $w=1$. From now on, we compute residues on both sides at $w=1$ stating the equality between them. Then the left-hand side turns out to be the sum in (4.1); it is a trace when we take the function $\Psi(s,r)$ as the Selberg transform. Thus the equality gives a new expression for a trace of Selberg type in terms of the theta series and the non-holomorphic Poincaré series(see Theorem 4.2).

The residue of the Eisenstein series $E(z,w,\Gamma)(\Gamma=PSL(2,\mathbf{Z}))$ at $w=1$ is $3/\pi$. Thus the residue of the left-hand side at $w=1$ becomes

$$\frac{12}{\pi}\sum_{j\geq 1}\Psi(s,r_j). \tag{4.1}$$

We next consider the first term on the right-hand side of (1.22). It may be rewritten as

$$\begin{aligned}&(4\pi)^{s-\frac{1}{2}}\Gamma(s)\int_{\Gamma_0(4)\backslash\mathcal{H}}\overline{\Theta(z)}P_1(z,s,\Gamma)\\ &\quad\times\Big\{2^w\pi^{\frac{w}{2}-1}\Gamma(\frac{w}{2}+\frac{1}{2})\frac{\zeta(w)}{2}\overline{E_{1/2}(z,\frac{\overline{w}}{2}+\frac{1}{2})}\Big\}\,d\mu(z).\end{aligned} \tag{4.2}$$

Recall the expansion (2.12) of $E_{1/2}(z,\frac{w}{2}+\frac{1}{2})$. In view of this we have

$$\begin{aligned}&2^w\pi^{\frac{w}{2}-1}\Gamma(\frac{w}{2}+\frac{1}{2})\frac{\zeta(w)}{2}\overline{E_{1/2}(z,\frac{\overline{w}}{2}+\frac{1}{2})}\\ &\quad=\zeta(w)\zeta_4(2w-1)B(z,w)\\ &\qquad+\zeta(w)\zeta_4(w)F(z,w)+\zeta(w)G(z,w),\end{aligned} \tag{4.3}$$

where

$$B(z,w)=2^{w-1}\pi^{\frac{w}{2}-1}\overline{i}^{1/2}2^{\frac{1}{2}-2w}\pi\frac{\Gamma(w-\frac{1}{2})}{\Gamma(\frac{w}{2})}\frac{1}{\zeta_4(2w)}2i\overline{z}^{1/2}Y^{1-\frac{w}{2}}, \tag{4.4}$$

$$\begin{aligned}F(z,w)=\;&2^{w-1}\pi^{\frac{w}{2}-1}\Gamma(\frac{w}{2}+\frac{1}{2})4^{-\frac{w+1}{2}}\frac{1}{\zeta_4(2w)}2i\overline{z}^{1/2}Y^{\frac{w+1}{2}}\overline{i}^{1/2}i^{-1/2}\\ &\times\sum_{m=1}^{\infty}\beta(-m^2,w+1)e^{2\pi im^2X}\tau_{-m^2}(Y,\frac{w}{2},\frac{w+1}{2}),\end{aligned} \tag{4.5}$$

$$\begin{aligned}G(z,w)=\;&2^{w-1}\pi^{\frac{w}{2}-1}\Gamma(\frac{w}{2}+\frac{1}{2})4^{-\frac{w+1}{2}}\frac{1}{\zeta_4(2w)}2i\overline{z}^{1/2}Y^{\frac{w+1}{2}}\overline{i}^{1/2}i^{-1/2}\\ &\times\sum_{t}\sum_{m=1}^{\infty}L_4(w,\omega_1)\beta(tm^2,w+1)e^{-2\pi itm^2X}\tau_{tm^2}(Y,\frac{w}{2},\frac{w+1}{2}),\end{aligned} \tag{4.6}$$

and $Y = y/4(x^2+y^2)$, $X = -x/4(x^2+y^2)$. Notice that the functions B, F and G are holomorphic at $w=1$.

The Laurent expansion of the Riemann zeta function at $w=1$ is $\zeta(w) = 1/(w-1)+\gamma_0+\cdots(\gamma_0$ is Euler's constant). Moreover since $\zeta_4(w) = (1-2^{-w})\zeta(w)$, we have

$$\begin{aligned}&\zeta(w)\zeta_4(2w-1)B(z,w) \\ &\quad= \frac{\frac{1}{4}B(z,1)}{(w-1)^2} + \frac{1}{w-1}\Big((c_0+\frac{1}{4}\gamma_0)B(z,1)+\frac{1}{4}B'(z,1)\Big)+\cdots,\end{aligned} \tag{4.7}$$

where $B'(z,w)=(d/dw)B(z,w)$ and

$$c_0 = \frac{1}{2}(\gamma_0+\log 2). \tag{4.8}$$

Similarly

$$\begin{aligned}&\zeta(w)\zeta_4(w)F(z,w) \\ &\quad= \frac{\frac{1}{2}F(z,1)}{(w-1)^2} + \frac{1}{w-1}\Big((c_0+\frac{1}{2}\gamma_0)F(z,1)+\frac{1}{2}F'(z,1)\Big)+\cdots.\end{aligned} \tag{4.9}$$

Therefore

$$\begin{aligned}&\zeta(w)\zeta_4(2w-1)B(z,w)+\zeta(w)\zeta_4(w)F(z,w) \\ &\quad= \frac{\frac{1}{2}}{(w-1)^2}\Big(\frac{1}{2}B(z,1)+F(z,1)\Big) \\ &\quad+\frac{1}{w-1}\Big\{(c_0+\frac{1}{2}\gamma_0)(\frac{1}{2}B(z,1)+F(z,1))-\frac{1}{2}c_0B(z,1)\Big\} \\ &\quad+\frac{1}{w-1}\Big(\frac{1}{4}B'(z,1)+\frac{1}{2}F'(z,1)\Big)+\cdots.\end{aligned} \tag{4.10}$$

Futhermore

$$\zeta(w)G(z,w) = \frac{1}{w-1}G(z,1)+\cdots. \tag{4.11}$$

We compute $\frac{1}{2}B(z,1)+F(z,1)$ in (4.10) explicitly. From (4.4)

$$\frac{1}{2}B(z,1) = 2^{-3/2}\pi^{1/2}\bar{i}^{1/2}\frac{1}{\zeta_4(2)}\frac{1}{2}2i\bar{z}^{1/2}Y^{\frac{1}{2}}.$$

Moreover since we can derive that

$$\tau_{-m^2}(Y,\frac{1}{2},1) = i^{1/2}2^{1/2}\pi Y^{-\frac{1}{2}}e^{-2\pi m^2 Y},$$

we obtain

$$F(z,1)=2^{-3/2}\pi^{1/2}\bar{i}^{1/2}\frac{1}{\zeta_4(2)}2i\bar{z}^{1/2}Y^{\frac{1}{2}}\sum_{m=1}^{\infty}\beta(-m^2,2)e^{2\pi i m^2 X}e^{-2\pi m^2 Y}.$$

Therefore noticing $\zeta_4(2)^{-1}=8/\pi^2$, we deduce

$$\begin{aligned}&\frac{1}{2}B(z,1)+F(z,1)\\&\quad=2^{3/2}\pi^{-3/2}\bar{i}^{1/2}2i\bar{z}^{1/2}Y^{\frac{1}{2}}\Big(\frac{1}{2}+\sum_{m=1}^{\infty}\beta(-m^2,2)e^{2\pi i m^2 X}e^{-2\pi m^2 Y}\Big).\end{aligned}\tag{4.12}$$

Put $m^2=2^{2l}m_0^2$ with an odd positive integer m_0 and an integer $l\geq 0$. Then from the definition (2.10) of $\beta(n,w+1)$, we see that

$$\beta(-m^2,2)=\sum_{d|m_0}\frac{1}{d}\sum_{a|d}\mu(a)=1.$$

Moreover since $Y=y/4(x^2+y^2)$ and $X=-x/4(x^2+y^2)$, we have

$$\frac{1}{2}+\sum_{m=1}^{\infty}\beta(-m^2,2)e^{2\pi i m^2 X}e^{-2\pi m^2 Y}=\frac{1}{2}\sum_{m=-\infty}^{\infty}e^{-2\pi i m^2/4z}.$$

Here in view of the Poisson summation formula we see that the last sum is equal to $2^{-1/2}\bar{i}^{1/2}z^{1/2}\Theta(z)$. Therefore

$$\frac{1}{2}+\sum_{m=1}^{\infty}\beta(-m^2,2)e^{2\pi i m^2 X}e^{-2\pi m^2 Y}=2^{-1/2}\bar{i}^{1/2}z^{1/2}\Theta(z).\tag{4.13}$$

We substitute this into (4.12), deriving

$$\frac{1}{2}B(z,1)+F(z,1)=2^2\pi^{-3/2}|z|y(\sigma_0 z)^{1/2}\Theta(z),\tag{4.14}$$

where $\sigma_0=\begin{pmatrix}&-2^{-1}\\2&\end{pmatrix}(Y=y(\sigma_0 z))$. Applying this result to (4.10), and gathering (4.2), (4.3) and (4.11) together we conclude that the Laurent expansion at $w=1$ of the first term on the right-hand side of (1.22) is described as

$$\begin{aligned}&\frac{1}{(w-1)^2}(4\pi)^{s-\frac{1}{2}}\Gamma(s)2\pi^{-3/2}\\&\quad\times\int_{\Gamma_0(4)\backslash\mathcal{H}}|z|y(\sigma_0 z)^{1/2}|\Theta(z)|^2P_1(z,s,\Gamma)\,d\mu(z)\end{aligned}\tag{4.15}$$

$$+\frac{1}{w-1}(4\pi)^{s-\frac{1}{2}}\Gamma(s)(c_0+\frac{1}{2}\gamma_0)$$
$$\times 2^2\pi^{-3/2}\int_{\Gamma_0(4)\backslash\mathcal{H}}|z|y(\sigma_0 z)^{1/2}|\Theta(z)|^2 P_1(z,s,\Gamma)\,d\mu(z)$$
$$+\frac{1}{w-1}(4\pi)^{s-\frac{1}{2}}\Gamma(s)\int_{\Gamma_0(4)\backslash\mathcal{H}}\overline{\Theta(z)}P_1(z,s,\Gamma)$$
$$\times\Big\{-\frac{1}{2}c_0B(z,1)+\frac{1}{4}B'(z,1)+\frac{1}{2}F'(z,1)+G(z,1)\Big\}\,d\mu(z)+\cdots$$

for $\Re(s)>1$.

The Laurent expansion at $w=1$ of the second term on the right-hand side of (1.22) is

$$-\frac{1}{w-1}2\pi\Gamma(s)\Gamma(s-1)\frac{1}{\zeta(2)}+\cdots \tag{4.16}$$
$$=-\frac{1}{w-1}\frac{12}{\pi}\Gamma(s)\Gamma(s-1)+\cdots.$$

Next though we omit the precise argument here we can derive that the Laurent expansion at $w=1$ of the third term on the right-hand side of (1.22) is expressed as

$$-\frac{1}{(w-1)^2}\frac{6}{\pi^2}\int_{-\infty}^{\infty}\Psi(s,r)\,dr \tag{4.17}$$
$$-\frac{1}{w-1}\Big\{\frac{12\gamma_0}{\pi^2}\int_{-\infty}^{\infty}\Psi(s,r)\,dr+\frac{1}{\pi}K'(1,s)\Big\}$$
$$+\frac{1}{w-1}\frac{3}{\pi}\Gamma^2(s-\frac{1}{2})+\cdots,$$

where $K'(w,s)=d/dw(K(w,s)$ and $K(w,s)=(1/\zeta(2w))J_{C_0}(w,s)$.

We are ready to state the results. First the poles of second order appearing in (4.15) and (4.17) cancel. Therefore we obtain the following

THEOREM 4.1. *For $\Re(s)>1$, we have*

$$(4\pi)^{s-\frac{1}{2}}\Gamma(s)2\pi^{-3/2}\int_{\Gamma_0(4)\backslash\mathcal{H}}|z|y(\sigma_0 z)^{1/2}|\Theta(z)|^2P_1(z,s,\Gamma)\,d\mu(z)$$
$$=\frac{6}{\pi^2}\int_{-\infty}^{\infty}\Psi(s,r)\,dr,$$

where $\sigma_0=\begin{pmatrix} & -2^{-1}\\ 2 & \end{pmatrix}$ and $\Psi(s,r)$ is that of (1.14).

Finally gathering the residues in (4.1), (4.15) through (4.17) we deduce the following trace formula.

THEOREM 4.2. *For $\Re(s)>1$, we have*

$$
\begin{aligned}
&\frac{12}{\pi}\sum_{j\geq 1}\Psi(s,r_j)\\
&\quad=(4\pi)^{s-\frac{1}{2}}\Gamma(s)(c_0+\frac{1}{2}\gamma_0)2^2\pi^{-3/2}\\
&\qquad\times\int_{\Gamma_0(4)\backslash\mathcal{H}}|z|y(\sigma_0 z)^{1/2}|\Theta(z)|^2P_1(z,s,\Gamma)\,d\mu(z)\\
&\quad+(4\pi)^{s-\frac{1}{2}}\Gamma(s)\int_{\Gamma_0(4)\backslash\mathcal{H}}\overline{\Theta(z)}P_1(z,s,\Gamma)\\
&\qquad\times\Big\{-\frac{1}{2}c_0B(z,1)+\frac{1}{4}B'(z,1)+\frac{1}{2}F'(z,1)+G(z,1)\Big\}\,d\mu(z)\\
&\qquad-\frac{12}{\pi}\Gamma(s)\Gamma(s-1)\\
&\quad-\frac{12\gamma_0}{\pi^2}\int_{-\infty}^{\infty}\Psi(s,r)\,dr-\frac{1}{\pi}K'(1,s)+\frac{3}{\pi}\Gamma^2(s-\frac{1}{2}),
\end{aligned}
$$

where γ_0 is Euler's constant, c_0 is that of (4.8)*, and the functions B, F, G and K are those of* (4.4) *through* (4.6) *and in* (4.17).

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